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# Concatenation of Scales Below 1 eV *

Alan Chodos†
*Center for Theoretical Physics*
*Department of Physics, Yale University, New Haven, CT 06520-8120*

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**Abstract**

There are (at least) four numbers of physical and cosmological significance, whose inferred values, when expressed in mass units, cluster in a window below 1 eV. There are: the neutrino mass, the neutrino chemical potential, the cosmological constant, and the size of two extra dimensions (if the fundamental scale of gravity is $1-10$ TeV). In this note, we imagine ways in which these four numbers could all be connected.

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* Dedicated to Kurt Haller on his seventieth birthday.

† Electronic address : `chodos@hepvms.physics.yale.edu`

Numerology is not much in fashion these days. The grand tradition of Kepler [1], Balmer [2], Eddington [3], and Dirac [4] has hardly any followers, and even when one arises [5], he is quickly shot down [6]. So it is with some trepidation that I want to bring up the fact that at least four dimensionful numbers, all potentially of great physical or cosmological significance, and all of which have been discussed recently in the literature, seem to cluster in the few decades just below one electron volt, which I shall loosely refer to as "the window".

I shall endeavor to present a framework within which it can be imagined that these four numbers might have something to do with each other. Most of this is based on work that I did recently with D. G. Caldi [7]; one component, in addition, is the subject of work in progress with Erich Poppitz [8].

The four numbers in question are:

1) the neutrino mass. What one seems to deduce from a combination of the solar and atmospheric neutrino data, with perhaps a dash of LSND thrown in, is that neutrino mass differences lie inside this window, although LSND prefers a rather higher value. The literature on this topic is so extensive [9] that it would be redundant for me to attempt to review it here.

2) the neutrino chemical potential [10]. Of course there are at least 3 flavors of neutrino, and their chemical potentials may differ. This quantity measures the difference between the cosmological density of neutrinos and anti-neutrinos of whatever species one is interested in. In fact, all of them may vanish, but in any event, there is a sum rule that suggests that if they are non-zero, at least some of them may lie inside the window of interest.

3) the cosmological constant. Recently, interest in a non-zero value of the cosmological constant has revived [11], spurred by measurements of an accelerating universe. Of course the acceleration may not be due to a strictly constant cosmological constant as demanded by classical general relativity–there are other ideas, such as "quintessence" [12] and the effects of conformal gravity [13]. But what is true is that the required energy density, expressed in mass units, falls squarely within the window.

4) The magnitude of a pair of extra dimensions. Recently, inspired by certain stringy ideas, it has been suggested [14] that the fundamental scale for gravity is not the Planck scale, but rather something in the range $1 - 10$ TeV (conveniently, and tantalizingly, just above the reach of current accelerators). To achieve this, one needs to imagine that whereas all the fields associated with the standard model are confined to a 3-brane (i.e. four-dimensional spacetime), gravity propagates in six dimensions, two of which are compactified. The size

of the compactified dimensions is determined by the requirement that gravity is coupled to ordinary matter with Newton's constant. This fixes the size to be about a millimeter (for 1 TeV) or a hundredth of that (for 10 TeV), placing the inverse size somewhere within the window.

What Caldi and I tried to do was to relate 1)-3) by hypothesizing a neutrino condensate in the universe. We had in mind not a chiral condensate of the type that would be familiar in the context of QCD, but rather a condensate of Cooper pairs. The advantages of this are twofold: first, the magnitude of such a condensate is tied directly to the size of the Fermi surface, which in turn is governed by the chemical potential; so 1) and 2) are immediately related. Second, unlike the case of a chiral condensate, as long as the chemical potential is non-zero, a pairing condensate will form in any channel with an attractive interaction, no matter how small.

Our first thought was to generate the condensate within the framework of the known neutrino interactions described by the standard model. One integrates out the $W$ and $Z$, and expresses this interaction in effective four-fermi form. Then one employs a mean field approximation to generate a set of gap equations, and one studies these to determine which are the attractive channels.

The simplest case is if one considers only one flavor of neutrino, coupled to itself. Since one is using the standard model, the neutrino is considered massless. The condensate itself, if any, is what will generate the neutrino mass.

In this simple case, the gap equation takes the form

$$B = \frac{-G^2}{\pi^2} \int_{-\Lambda}^{\Lambda} dp\, p^2 \, \frac{B}{\sqrt{(p-\mu)^2 + 8MG^2}} \ . \qquad (1)$$

Here $B$ is the value of $\langle\psi\psi\rangle$, (with spinor indices suitably contracted), where $\langle\ \rangle$ denotes a vacuum expectation value, and $M = B^\dagger B$. $G^2$ is the effective four-fermi coupling, with dimension of inverse mass squared, which in the standard model is related to the usual fermi coupling constant ($10^{-5} M_p^{-2}$) by a coefficient of order one. The parameter $\mu$ is the chemical potential, and $\Lambda$ is the cutoff, which is necessary since the effective four-fermi theory is not renormalizable.

The important point to notice is that the two sides of this equation are of opposite sign; this implies a repulsive channel, and there is no solution to the gap equation other than the trivial one $B = 0$.

It is not surprising that this channel is repulsive. After all, the interaction is due to vector exchange, and as one learns from electrodynamics, like particles repel. Since we

have allowed only one species of particle, it only interacts with itself, and hence produces a repulsive interaction.

The next thing one might try is to introduce a variety of different flavors. The condensate will then have a matrix structure in flavor space:

$$B_{ij} = \langle \psi_i \psi_j \rangle \ , \tag{2}$$

and one might hope that some of the elements of this matrix would effectively represent attractive channels. But as long as the chemical potential remains flavor independent, one can show with a little group theory that the problem essentially factorizes into a flavor part times a spinor part, and the negative sign that was found for a single flavor persists. It is an interesting question to see what might happen for the case of a flavor dependent chemical potential, but Caldi and I did not examine that case, although we did speculate that flavor dependent chemical potentials might induce condensation in such a way as to generate an interesting spectrum of neutrino masses and mixings.

As yet another, more radical possibility, one might consider the pairing of neutrinos not with each other, but rather with charged leptons. Phenomenologically, this possibility would be rather drastic, because if such a condensate existed it would imply that the vacuum was not an eigenstate of charge. In any case, one finds that the pairing of a neutrino with a lepton of the same flavor remains repulsive, but the pairing of a neutrino with a lepton of different flavor (e.g. the muon neutrino with the electron) is attractive in the standard model. Because the two members of the pair have different mass, and because one must also allow different chemical potentials for the neutrino and the charged lepton, one obtains a rather more complicated form for the gap equation. Explicitly, one finds

$$-\kappa^2 B^\dagger B = \frac{-i\kappa^4 B^\dagger B}{\pi^3} \int_{-\Lambda}^{\Lambda} dp \ p^2 \int_{-\infty}^{\infty} dp_0 \mathcal{G}(p_0, p) \tag{3}$$

where

$$\mathcal{G}(p_0, p) = \frac{p_0 - \mu_e + p}{[(p_0 - \mu_e)^2 - p^2 - m^2][p_0 + \mu'_\nu + p] - 4\kappa^4 B^+ B[p_0 - \mu_e + p]} \ . \tag{4}$$

Here $B = \langle \nu_\mu \gamma^0 \gamma^2 e \rangle$ (a Lorentz scalar), and $\kappa^2 = G^2(1 - 2sin^2\theta_W)$, where $\theta_W$ is the Weinberg angle. The parameters $\mu_e$ and $\mu'_\nu$ are the electron and neutrino chemical potentials, respectively (the prime is to remind us that the neutrino carries a different flavor quantum number from the charged lepton).

It is not possible to simplify this expression much in the general case. But it is instructive to look at some special limits, which I do in the appendix. In particular, one learns that

the property of admitting a solution for arbitrarily weak attraction is maintained only if the lepton and neutrino densities are equal and non-vanishing, which is certainly not true of the present-day universe.

Caldi and I also noted that there was the possibility of lepton-neutrino condensation in a Lorentz-non-invariant channel, but we did not investigate this.

The unrenormalized gap equations that I have introduced so far are not suited to determining the overall size of the condensate, because this will depend on the cutoff. Since the underlying theory (the standard model) is renormalizable, presumably the cutoff dependence can be consistently eliminated by introducing renormalized parameters. One then expects that the expression for the condensate will take a BCS-like form:

$$\triangle = \kappa^2 B^\dagger B \sim \mu e^{-\frac{1}{\mu^2 G^2}} \; . \tag{5}$$

The condensate is proportional to the chemical potential, as expected, but since the generation of a gap is a non-perturbative phenomenon, one has the characteristic non-analytic dependence on the coupling exhibited in eq. (2). The exponential reflects the fact that as the gap tends to zero, the gap equation exhibits a logarithmic singularity at the fermi surface. In our case, this exponential factor is extremely small, pushing the condensate far outside the desired window. This led Caldi and me to hypothesize the existence of a new interaction, acting only among neutrinos, which would effectively generate a four-fermi coupling with a scale in the range between 1 eV and $10^{-2}$ eV, instead of the 250 GeV given by the standard model. Furthermore, of course, we assumed the interaction to be attractive, effectively eliminating the sign problem in eq. (1). Granting that this is ad hoc, let us just pursue it a bit to see how the various quantities with which we began enter the discussion.

Let us suppose that the dynamics of the neutrinos is governed by an effective attractive four-fermi interaction of the form (in two-component notation)

$$\mathcal{L}_{int} = -2G^2 (\psi^\dagger_\alpha \epsilon_{\alpha\gamma} \psi^\dagger_\gamma) \; (\psi_\beta \epsilon_{\beta\delta} \psi_\delta) \tag{6}$$

where the coupling G is of the order an inverse eV. Furthermore, we assume that there is a sufficiently large chemical potential to induce condensation of neutrino pairs with a scale of an eV. This will immediately give rise to a Majorana neutrino mass of magnitude

$$m = 2G^2 \mid \langle \psi^\dagger_\alpha \epsilon_{\alpha\gamma} \psi^\dagger_\gamma \rangle \mid \; . \tag{7}$$

Furthermore, the condensate represents a contribution to the vacuum energy density:

$$\epsilon_0 = -2G^2 \langle \psi^\dagger_\alpha \epsilon_{\alpha\gamma} \psi^\dagger_\gamma \rangle \langle \psi_\beta \epsilon_{\beta\delta} \psi_\delta \rangle \ . \tag{8}$$

According to the usual lore, however, this energy density will represent an effective cosmological constant only if the system is not in the true vacuum, but is rather in a supercooled state or perhaps in a very slowly rolling state in which the size of the condensate is decaying toward zero (the true vacuum). For definiteness, let us imagine the former possibility. Then at some earlier time in the history of the universe, the neutrinos condensed, and for a while the universe was in the true ground state. As the universe evolved, the state with the condensate present ceased to be the true ground state, so there can exist an inflationary epoch (with a rather small cosmological constant) that persists until the universe undergoes another phase transition to the true vacuum.

Does the above scenario make any sense when confronted with the standard picture of how the universe is evolving? In the earlier epoch, the background neutrinos were both at a higher density and a higher temperature. As a consequence of its expansion, the universe traces out a particular path as a function of time in the neutrino temperature/density plane, heading for lower values of both variables as time goes on. According to the standard big-bang cosmology, $T \sim \frac{1}{R(t)}$ and $\rho \sim \frac{1}{R^3(t)}$, so the curve in question is just $\rho \propto T^3$. Now the Cooper pairing phase we have been discussing is a low-temperature high-density phase. If the universe enters this phase at all, one expects that it will eventually emerge from it as the density drops. There is a program underway capable of testing this expectation, in a one-plus-one dimensional simulation. This is based on a recently introduced model [15] similar to the Gross-Neveu model [16], which, however, also contains a Cooper-pair type condensate as well as the usual chiral condensate. Once the phase diagram of this model is understood, it will be possible to trace out the history of the universe, to see whether there exists an epoch such that the system is in a false vacuum characterized by a neutrino condensate that will eventually disappear, along with the attendant cosmological constant.

To summarize thus far: the formation of a cosmological condensate of Cooper pairs would appear to require some new interaction acting only among neutrinos. If this interaction is attractive and has the right magnitude, and if the relevant chemical potentials are as large as are allowed by the sum rules [10], and if we are at present in a supercooled state, then both the neutrino mass and the cosmological constant can be viewed as dynamical consequences of this condensation phenomenon.

The reader may have been remarking at the high dosage of speculation that has been necessary to get us to this point; however, the most speculative part of this discussion is yet to come. We shall adopt a recent idea [14] motivated by certain aspects of string theory: that

the fundamental scale of quantum gravity is not the Planck scale, but rather something not much bigger than a TeV. If this is true, it opens up the exciting possibility that quantum-gravitational effects will be accessible to the next generation of accelerators. The price to be paid, however, is that gravity does not propagate in four dimensions, but rather in a six-dimensional spacetime, two of whose dimensions are compactified into a manifold with the characteristic size of a millimeter.

Now a millimeter is equivalent to about $10^{-4}$ eV, so this is yet another scale that falls within our window. Is there any way of tying it to the other three scales we have been discussing so far? We begin by noting [8, 17] that there exists an exact classical solution of the six-dimensional Einstein equations with a cosmological constant $\lambda_0$, which is the product of a four-dimensional de Sitter space and a static 2-sphere. If the equation defining the de Sitter space is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \alpha g_{\mu\nu} = 0 \ , \tag{9}$$

then the cosmological constant $\alpha$ is related to the bulk cosmological constant by

$$\lambda_0 = 2\alpha \tag{10}$$

and to the radius of the 2-sphere $R_0$ simply by

$$\alpha = \frac{1}{R_0^2} \ . \tag{11}$$

However, the relationship between the energy density associated with the observed cosmological constant and the postulated size of the extra dimensions involves a power of the Planck scale, so this relationship is violated by many orders of magnitude in nature (about 27). Work is currently in progress [8] to find other solutions of the six-dimensional equations, which include not only gravity but also, at least partially, the effects of the matter that resides on the four-dimensional “3-brane” that we know as spacetime, and which will involve so-called “warp factors” as well. It remains to be seen whether such modifications can lead to a realistic geometry that incorporates both the observed cosmological constant and a millimeter-sized pair of extra dimensions.

The purpose of this article has been to weave some scenarios, hopefully not too far outside the bounds of credibility, that might begin to explain the remarkable concatenation of scales that one observes in the mass region just below 1 eV. The two ideas that we have concentrated upon are, first, the existence of a cosmological neutrino condensate of the Cooper pairing variety, and, second, a cosmology governed by a geometry more complicated than but not too unlike the one described in eqns. (9-11).

## Appendix

Here we examine special limits of the gap equation given by eqns. (3) and (4). In particular, we shall briefly discuss the following: (i) $m = 0$; (ii) $\mu_e = -\mu'_\nu = \mu$; and (iii) $B^\dagger B = 0$.

(i) When $m = 0$, we are considering the pairing of two massless leptons. We therefore expect something similar to the gap equation of eq. (1); the only difference is that we are still allowing them to have unequal chemical potentials. We find that in this limit, $\mathcal{G}_0$ simplifies to:

$$\mathcal{G}_0(p_0, p) = \frac{1}{(p_0 - \mu_e - p)(p_0 + \mu'_\nu + p) - \Delta^2} \tag{12}$$

where $\Delta^2 = 4\kappa^2 B^\dagger B$. The $p_0$ integral can then be done, giving for the gap equation

$$1 = \frac{\kappa^2}{\pi^2} \int_{-\Lambda}^{\Lambda} dp\, p^2 \frac{\theta[(p + \frac{\mu_e+\mu'_\nu}{2})^2 + \Delta^2 - (\frac{\mu_e-\mu'_\nu}{2})^2]}{[(p + (\frac{\mu_e+\mu'_\nu}{2}))^2 + \Delta^2]^{\frac{1}{2}}} \ . \tag{13}$$

The $\theta$-function in the numerator arises from the $i\epsilon$ prescription $p_0 \to p_0 + i\epsilon sgn p_0$ that is implicit in the expression for $\mathcal{G}_0$. We observe that, first, this is an attractive channel since both sides of the gap equation have the same sign. Second, without the $\theta$-function (and modulo an inconsequential change in the overall sign of the chemical potential) this is of the same general form as equation (1). Third, the role of the $\theta$-function is to protect the denominator from vanishing in the range of integration in the limit $\Delta \to 0$. As $\mu'_\nu - \mu_e \to 0$, this protection disappears, and one regains the instability at the Fermi surface that one sees in equation (1).

(ii) When $\mu'_\nu = -\mu_e = -\mu$, $\mathcal{G}_0$ becomes

$$\mathcal{G}_0(p_0, p) = \frac{1}{(p_0 - \mu)^2 - p^2 - m^2 - \Delta^2} \ . \tag{14}$$

In this case, we see immediately that $\Delta^2$ acts like an extra contribution to $m^2$, and doing the $p_0$ integral we obtain

$$1 = \frac{\kappa^2}{\pi^2} \int_{-\Lambda}^{\Lambda} dp\, p^2 \frac{\theta(p^2 + m^2 + \Delta^2 - \mu^2)}{\sqrt{p^2 + m^2 + \Delta^2}} \ . \tag{15}$$

Here there is no possibility of an infrared instability, and one will in general have a solution only if $\kappa$ is large enough. This is exactly like a typical gap equation associated with a chiral

condensate, which is not surprising since by setting $\mu'_\nu = -\mu_e$ we are effectively treating one member of the pair as a particle and the other as an anti-particle.

(iii) Finally, we look at $\Delta = 0$. Of course, in setting $\Delta = 0$ we can no longer regard the equation as a gap equation telling us how large $\Delta$ ought to be. What we can do, however, is first, verify that for general $m$, $\mu_e$ and $\mu'_\nu$ the channel is attractive, and second, discover under what restrictions on $m, \mu_e$ and $\mu'_\nu$ we can obtain a pairing instability. In this case,

$$\mathcal{G}_0(p_0, p) = \frac{p_0 - \mu_e + p}{[(p_0 - \mu_e)^2 - p^2 - m^2][p_0 + \mu'_\nu + p]} \tag{16}$$

and doing the $p_0$ integral we find, with $\omega = \sqrt{p^2 + m^2}$,

$$1 = \frac{\kappa^2}{\pi^2} \int_{-\Lambda}^{\Lambda} dp \; p^2 \{ \frac{\theta(\omega^2 - \mu_e^2)}{\omega} [\frac{(\omega - p)\theta(-p - \mu'_\nu)}{\omega - p - \mu_e - \mu'_\nu} + \frac{(\omega + p)\theta(p + \mu'_\nu)}{\omega + p + \mu_e + \mu'_\nu}]$$
$$+ \frac{2(\mu_e + \mu'_\nu)}{(\omega + p + \mu_e + \mu'_\nu)(\omega - p - \mu_e - \mu'_\nu)} [\theta(-\mu_e - \omega)\theta(-p - \mu'_\nu) - \theta(\mu_e + \omega)\theta(p + \mu'_\nu)]\} \; . \tag{17}$$

Careful consideration of the $\theta$-functions reveals that the integrand is always positive, thereby assuring us that the channel is always attractive. The only conditions under which a denominator can vanish, however, are

$$p^2 = \mu'^2_\nu = \mu_e^2 - m^2 \; . \tag{18}$$

This pair of conditions tells us that the fermi momenta of the two members of the pair must coincide in order for an instability to develop.

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